On the density of nearly regular graphs with a good edge-labelling

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Abstract

A good edge-labelling of a simple graph is a labelling of its edges with real numbers such that, for any ordered pair of vertices (u, v), there is at most one nondecreasing path from u to v. Say a graph is good if it admits a good edge-labelling, and is bad otherwise. Our main result is that any good n-vertex graph whose maximum degree is within a constant factor of its average degree (in particular, any good regular graph) has at most $n^{1+o(1)}$ edges. As a corollary, we show that there are bad graphs with arbitrarily large girth, answering a question of Bode, Farzad and Theis. We also prove that for any Δ, there is a g such that any graph with maximum degree at most Δ and girth at least g is good.

1 Introduction

A *good edge-labelling* of a simple graph is a labelling of its edges with real numbers such that, for any ordered pair of vertices (u, v), there is at most one nondecreasing path from u to v. This notion was introduced in [3] to solve wavelength assignment problems for specific categories of graphs. Say graph G is *good* if it admits a good edge-labelling, and is *bad* otherwise.

Let $\gamma(n)$ be the maximum number of edges of a good graph on n vertices. Araújo, Cohen, Giroire, and Havet [2] initiated the study of this function. They observed that hypercube graphs are good, and any graph containing K_3 or $K_{2,3}$ is bad, thus

$$\Omega(n \log n) \leq \gamma(n) \leq O(n\sqrt{n}).$$

Our main result is that any good graph whose maximum degree is within a constant factor of its average degree (in particular, any good regular graph) has at most $n^{1+o(1)}$ edges. Until now, no bad graphs with girth larger than 4 were known [2, 4]. Bode, Farzad and Theis [4] asked whether all graphs with large enough girth are good. As a corollary of our main result, we give a negative answer by proving that there are bad graphs with arbitrarily large girth. We also give a very short proof that the answer is positive for bounded degree graphs.

2 The Proofs

For a graph G and an edge-labelling $\phi : E(G) \to \mathbb{R}$, a *nice k-walk from v_0 to v_k* is a sequence $v_0 v_1 \ldots v_k$ of vertices such that $v_{i-1}v_i$ is an edge for $1 \leq i \leq k$, and $v_{i-1} \neq v_{i+1}$ and $\phi(v_{i-1}v_i) \leq$

$\phi(v_i v_{i+1})$ for $1 \leq i \leq k-1$. The existence of a self intersecting nice walk implies that the edge-labelling is not good: let $v_0 v_1 \ldots v_k$ be a shortest such walk with $v_0 = v_k$. Then there are two nondecreasing paths $v_0 v_1 \ldots v_{k-1}$ and $v_0 v_{k-1}$ from v_0 to v_{k-1}. Thus if for some pair of vertices (u, v) there are two nice k-walks from u to v, then the labelling is not good.

Let $f_k(n, m, \Delta)$ be the maximum number f such that every edge-labelling of a graph on n vertices, at least m edges and maximum degree at most Δ, has at least f nice k-walks.

Lemma 1. *Let n, m, Δ, k, a be positive integers with $k > 1$ and $a \leq \Delta/2$. We have $f_1(n, m, \Delta) = m$ and*

$$f_k(n, m, \Delta) \geq a \left[f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m) a (\Delta - a)^{k-3} \right].$$

Proof. Since any edge is a nice 1-walk, we have $f_1(n, m, \Delta) = m$. Let G be a graph with n vertices, at least m edges, and maximum degree at most Δ. Call a vertex of G *wealthy* if its degree is larger than a, and *beggared* otherwise. Let b the number of beggared vertices. Since every wealthy vertex has degree at most Δ, and the sum of degrees is at least $2m$, we have

$$ba + (n - b)\Delta \geq 2m,$$

so $b \leq (n\Delta - 2m)/(\Delta - a)$.

Let v be a wealthy vertex and $e_1, \ldots, e_d$ be its incident edges, ordered such that

$$\phi(e_1) \geq \phi(e_2) \geq \cdots \geq \phi(e_d).$$

Call the edges $e_1, e_2, \ldots, e_a$ the *strong* edges for v. Let S be the set of all strong edges for all wealthy vertices. Clearly $|S| \leq na$. Let H be the graph obtained from G by deleting the edges in S. Note that H has n vertices, at least $m - an$ edges, and maximum degree at most $\max\{a, \Delta - a\} = \Delta - a$.

For a wealthy vertex v, every nice $(k-1)$-walk in H ending in v can be extended to a distinct nice k-walks in G. Thus every nice $(k-1)$-walk in H whose both endpoints are wealthy, can be extended to a distinct nice k-walks in G. By definition, there are at least $f_{k-1}(n, m - an, \Delta - a)$ nice $(k-1)$-walks in H. The number of $(k-1)$-walks in H starting from a beggared vertex is not more than

$$ba(\Delta - a)^{k-2} \leq (n\Delta - 2m) a (\Delta - a)^{k-3},$$

since there are b choices for the first vertex, at most a choices for the second vertex, and at most $\Delta - a$ choices for the other $k - 2$ vertices. Hence there are at least

$$f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m) a (\Delta - a)^{k-3}$$

nice $(k-1)$-walks in H whose both endpoints are wealthy, and the lemma follows. $\square$

Let $q \in (0, 1/2)$ be a fixed number that will be determined later, and let $p = 1 - q$. Setting $a = q\Delta$ in the lemma gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3} \Delta^{k-1} (n\Delta - 2m),$$

provided that $q\Delta$ is an integer.

Define two sequences $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ by $a_1 = 1, b_1 = 0$, and for $k > 1$,

$$\begin{aligned} a_k =& qp^{k-2}a_{k-1} + 2q^2p^{k-3} \\ b_k =& q^2p^{k-2}a_{k-1} + qp^{k-1}b_{k-1} + q^2p^{k-3}, \end{aligned}$$

And define the function $g_k(n, m, \Delta)$ as

$$g_k(n, m, \Delta) = a_k m\Delta^{k-1} - b_k n\Delta^k.$$

One computes $g_1(n, m, \Delta) = m$ and

$$g_k(n, m, \Delta) = q\Delta g_{k-1}(n, m - qn\Delta, p\Delta) - q^2p^{k-3}\Delta^{k-1}(n\Delta - 2m).$$

Hence

$$f_1(n, m, \Delta) = g_1(n, m, \Delta),$$

and it is easy to show by induction on k that given t,

$$f_k(n, m, \Delta) \geq g_k(n, m, \Delta), \tag{1}$$

for $1 \leq k \leq t$, provided that $q\Delta, qp\Delta, \ldots, qp^{t-2}\Delta$ are positive integers.

Lemma 2. *For any positive integers t and c, if q is sufficiently small then $a_t > cb_t$.*

Proof. Define $x_k = a_k/q^{k-1}$ and $y_k = b_k/q^{k-1}$. Then

$$\begin{aligned} &x_1 = 1, y_1 = 0, \\ &x_k = p^{k-2}x_{k-1} + 2qp^{k-3}, \\ &y_k = qp^{k-2}x_{k-1} + p^{k-1}y_{k-1} + qp^{k-3}. \end{aligned}$$

Clearly, $a_t > cb_t$ if and only if $x_t > cy_t$. Note that since $p = 1 - q < 1$, we have $x_k \leq x_{k-1} + 2q$. Assume that $q < 1/2t$. So $x_k \leq 2$ for all $1 \leq k \leq t$.

Now let $z_k = x_k - cy_k$. Then $z_1 = 1$ and

$$z_k = p^{k-2}\left(x_{k-1} - cpy_{k-1}\right) + qp^{k-3}(2 - c) - cqp^{k-2}x_{k-1}.$$

Note that $p < 1$, $x_k \leq 2$ and $y_k \geq 0$ for all $1 \leq k \leq t$, so for k in this range,

$$z_k \geq p^{k-2}z_{k-1} - 3cq.$$

Hence,

$$z_t \geq p^{t-2}p^{t-3}\ldots p^2p - 3cq(t-1) \geq p^{t^2/2} - 3cqt.$$

Define $h(q) := (1-q)^{t^2/2} - 3cqt$. Since $h(0) = 1$ and h is continuous, there is a $q_0 > 0$ such that $h(q) > 0$ for all $0 \leq q < q_0$. So for $0 < q < \min\{\frac{1}{2t}, q_0\}$ we have $a_t > cb_t$. □

Now we prove our main result, which states that any good graph whose maximum degree is within a constant factor of its average degree (in particular, any good regular graph) has average degree $n^{o(1)}$. For a graph G, denote its maximum degree and average degree by $\Delta(G)$ and $\overline{d}(G)$, respectively.

Theorem 3. *For any positive integers t and c there is an $\epsilon(t,c) > 0$ such that any n-vertex graph G with $\Delta(G) \le c\overline{d}(G)$ and $\epsilon(t,c)\overline{d}(G)^t > n$ is bad.*

Proof. Let q' be a large enough integer so that for $q = 2^{-q'}$, $a_t - 4cb_t > 0$. Let $q = 2^{-q'}$ and $\alpha_t = \frac{a_t}{4} - cb_t > 0$. We claim that $\epsilon(t,c) = \min\{c^{t-1}\alpha_t, 2^{-q't^2}\}$ works.

Let G be an n-vertex graph with $\Delta(G) \le c\overline{d}(G)$ and $\epsilon(t,c)\overline{d}(G)^t > n$. Let $\overline{d} = \overline{d}(G)$ and $r = 2^{r'}$, where $r' = \lceil \log_2 \overline{d} \rceil$, so $r/2 < \overline{d} \le r$. We have

$$2^{-q't^2} r^t \ge \epsilon(t,c)\overline{d}^t > n \ge 1,$$

so $r > 2^{q't}$ and thus $qcr, qpcr, \dots, qp^{t-2}cr$ are positive integers. Hence (1) with $m = \frac{nr}{4}$ and $\Delta = cr$ holds for $1 \le k \le t$ and thus

$$f_t\left(n, \frac{nr}{4}, cr\right) \ge g_t\left(n, \frac{nr}{4}, cr\right) = a_t\left(\frac{nr}{4}\right)(cr)^{t-1} - b_t n (cr)^t = nr^t c^{t-1}\alpha_t \ge n\overline{d}^t \epsilon(t,c) > n^2.$$

Let ϕ be any edge-labelling of G. Note that G has at least $nr/4$ edges and maximum degree at most cr, so $f_t(n, nr/4, cr) > n^2$ means that G has more than n^2 nice t-walks. By the pigeonhole principle, there is an ordered pair of vertices (u,v) such that there are two distinct nice t-walks from u to v, hence the labelling is not good. $\square$

Corollary 4. *For any integer $g \ge 3$ there is a bad graph with girth g.*

Proof. Since K_3 and $K_{2,3}$ are bad, we may assume that $g \ge 5$. Let t be a positive integer larger than $3g/4$, and let d be an odd prime power larger than $2/\epsilon(t,1)$. Lazebnik, Ustimenko and Woldar [5] proved that there is a d-regular graph G with girth g with at most $2d^{\frac{3}{4}g-1}$ vertices. So

$$|V(G)| \le 2d^{\frac{3}{4}g-1} < \epsilon(t,1)d^t,$$

and G is bad by Theorem 3. $\square$

Next we show that for any Δ, there is a $g = g(\Delta)$ such that any graph with maximum degree at most Δ and girth at least g is good.

Theorem 5. *Let G be a graph with girth at least $2k$ and maximum degree at most Δ. If*

$$4ek^2(\Delta-1)^{k-1} < k!$$

then G admits a good edge-labelling.

Proof. Choose the label of each edge independently and uniformly at random from the interval $[0,1]$. If the labelling is not good, then since the graph has girth at least $2k$, there must exist a nondecreasing path of length exactly k. For any path of length k, the probability that it is a nondecreasing path is $2/k!$. Moreover, every path of length k intersects at most $2k^2(\Delta-1)^{k-1} - 1$ other paths of length k. Hence by the Lovász Local Lemma (see, e.g., Chapter 5 of [1]) there is a positive probability that the edge-labelling is good, and the proof is complete. $\square$

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References

[1] N. Alon and J. Spencer, The Probabilistic Method, 3rd edition. John Wiley & Sons, New York, 2008.

[2] J. Araújo, N. Cohen, F. Giroire, and F. Havet. Good edge-labelling of graphs. Discrete Applied Mathematics, to appear.

[3] J.-C. Bermond, M. Cosnard, and S. Pérennes. Directed acyclic graphs with unique path property. Technical Report RR-6932, INRIA, May 2009.

[4] M. Bode, B. Farzad, and D. O. Theis. Good edge-labelings and graphs with girth at least five. preprint, 2011. available on arXiv:1109.1125.

[5] F. Lazebnik, V. A. Ustimenko, A. J. Woldar. New upper bounds on the order of cages. Electronic Journal of Combinatorics 4 (1997), no. 2, Research Paper 13.